Technical report #9

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Introduction

Number of solutions and symmetry positions for tilings on a strip

abstract

The number of distinct ways to transform between two geometrical structures which are symmetric to each other is defined to be equal to the number of distinct symmetrical positions at half-way. Though this work only a most simple structure of tilings on a strip is considered, the same idea can be applied to other more complex structures.

Theory

Assumption 1 *Relevant geometrical structures are those which can be transformed from one of the two opposite positions to the other.*

An example of such structures is given in Figure 1. Here the *opposite positions* can be thought of in two ways depending on whether the colour or the insidedness is being considered. For the former one may say that the two opposite positions are the one which the inside is white as opposite to the one which the inside is grey. For the latter one may say that they are the one which the grey colour is inside as opposite to the one which the grey colour is outside.

Lemma 1 *There exists a specific number of distinct half-way points.*

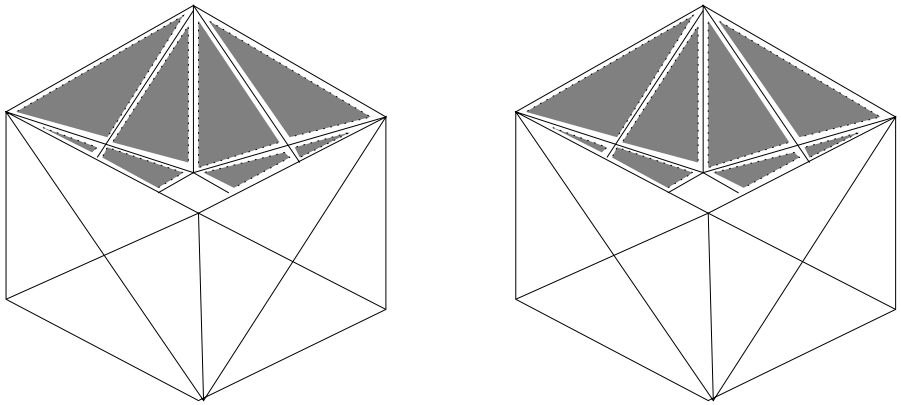

Figure 1: The two opposite positions are; the one in which the grey coloured sides of the tiles are all on the outside (a) and; the one in which they are all on the inside (b).

Proof
Suppose that the opposite is true that is, there are infinite number of such points. It is obvious that the number of half-way points is $\leq$ the number of all possible positions $= \infty$. But the number of positions must be finite since it is the combinations of finite number of geometrical structures with definite constraints of relative movement. Thus the contradiction and, therefore the original statement has to be true. QED

Definition 1 *The number of unique solutions is equal to the number of distinct half-way positions.*

Then from Lemma 1 and Definition 1 the following lemma follows.

Lemma 2 *There exists a definite number of solutions. This number is equal to the number of distinct positions which are at half-way points between the two opposite positions.*

The following lemma deals with the nature of the half-way points. Here $\mathrm{Rev}(\cdot)$ is a *reversing function* which, in the case of the tile strip, represents toggling of the colour on both the inside and the outside at the same time. $\mathrm{Mir}(\cdot)$ is a *mirroring function* which gives the mirror image of a given structure. $P(x \to y)$ is a process which transform x to y where, x, y are positions.

Lemma 3 *Let M be a point at exactly half way going from a position A to its opposite position $B =$ $\mathrm{Rev}(A)$. Then*

$$M = \mathrm{Mir}(\,\mathrm{Rev}(M)) = \mathrm{Rev}(\,\mathrm{Mir}(M)).$$

Proof
Let A' be a position which is very close to M and is on the direction coming from A. Let B' be a position which is very close to M but is on the direction coming from B. Both the transition from A' to B' and that from B' to A' passes through M and both must be smooth at M. And each of the two transitions must be in exactly the opposite direction of the other that is, going from A' to B' should require the same amount of work as going from B' to A'. Moreover, going from A' to M and

going from M to A' coming from B' must be exactly the same job going in the opposite direction. In effect, this is saying that $P(A' \to B')$ is exactly the same job as $P(B' \to A')$ but not identical ones. Since both of the two processes is being done on the same structure this is only possible if they are mirror images to each other. That is

$$P(A' \to B') = \text{Mir}(P(\text{Rev}(B' \to A'))).$$

When $A' \to M$ and $B' \to M$ this statement becomes the statement of the lemma. That is at the half-way point M and $\text{Rev}(M)$ must be a mirror image of each other. QED

By similar reasoning the following lemma follows.

Lemma 4

$$P(A \to B) = \text{Mir}(P(\text{Rev}(B \to A)))$$

Example of tessellations on a strip

The structure shown in Figure 1 is that of rigid tessellations on a flexible strip. Here A is the position where the strip is in normal position with all the tiles turning their grey faces outward and, B is that position where they turn them inward. The details of $P(A \to B)$ and $P(B \to A)$ is given in the appendix.

Conclusion

For geometrical puzzles such that Assumption 1 is met, the number of unique ways to solve them is definitive if Definition 1 is followed.

Details of the solution using both methods

Notice that Lemma 3 and Lemma 4 holds for both Method 1 (Figure 2) and Method 2 shown in (Figure 3) while for Method 1 it is also true that

$$M = \text{Mir}(M) = \text{Rev}(M),$$

and

$$P(A \to B) = P(\text{Rev}(B \to A)).$$

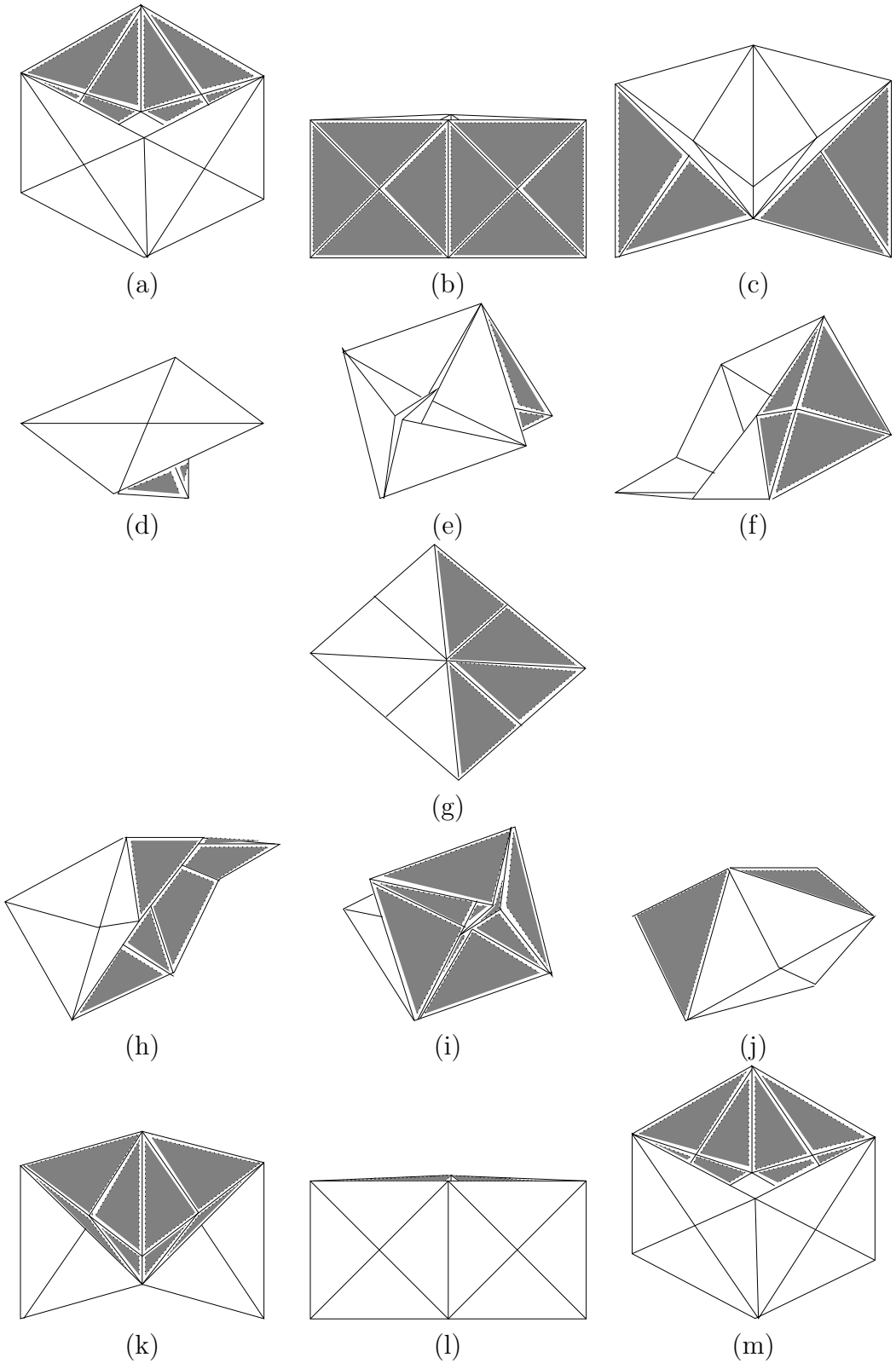

Figure 2: *Method 1, the symmetry at half-way is (g).*

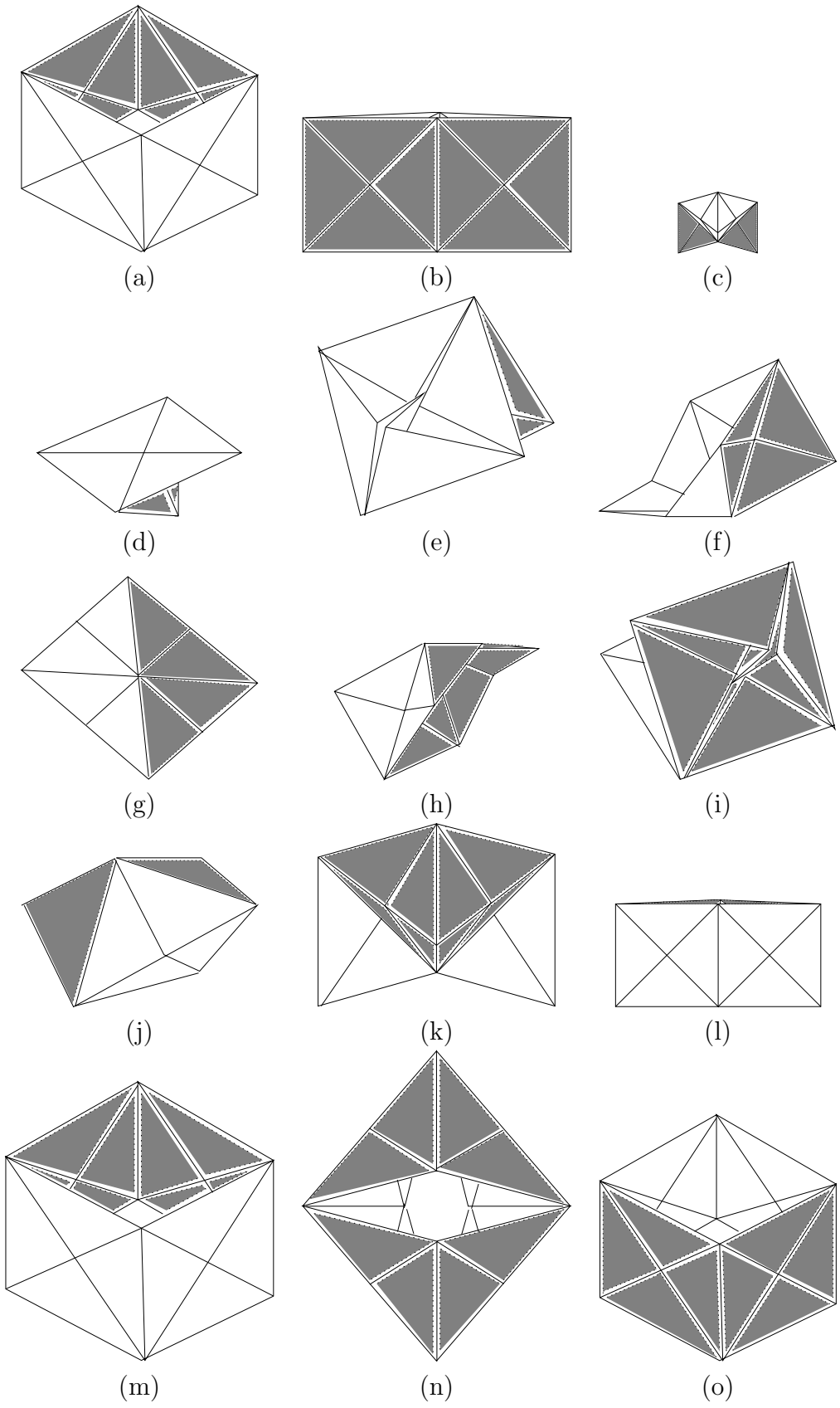

Figure 3: *Method 2, the symmetry at half-way is (h).*